Kuwait University
Department of Mathematics and Computer Science

Math 102 : Calculus II	Summer Semester 2009	In-Term Test 2

Saturday 1 August 2009

Duration: 90 minutes Total marks: 25

Justify all your answers

1. Find $\lim_{x\to 0^+} \left(e^x - 2\arctan x\right)^{1/x}$. [3 marks]

2. Find $\lim_{x\to\infty} (\sinh 2x - e^x \cosh x)$. [3 marks]

3. What is the form of the partial fraction decomposition of $\dfrac{2x^3+8x-1}{x^3+x^2-x-1}$? [1 mark]

4. Evaluate the following. [3 marks each]

   (a) $\displaystyle\int x\left(\sec^{-1}\sqrt{x}\right)\,dx$

   (b) $\displaystyle\int \left(\tan^3 x \sec^5 x - \tan^2 x \sin x + 4\sin^2 x\right)\,dx$

   (c) $\displaystyle\int \frac{2x-1}{\sqrt{x^2-6x+5}}\,dx$

   (d) $\displaystyle\int \frac{x^2+14}{(x-2)(x^2+2x+10)}\,dx$

   (e) $\displaystyle\int \frac{1}{x\left(\sqrt{\sqrt[6]{x}-1}\right)}\,dx$

   (f) $\displaystyle\int \frac{1+2\tan(x/2)}{2+\sin x}\,dx$

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ANSWERS

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1. The limit is of indeterminate type 1^∞. So consider the limit of the natural logarithm of the function, i.e.

$$\lim_{x\to0+}\ln[(e^x-2\arctan x)^{1/x}]=\lim_{x\to0+}\frac{\ln(e^x-2\arctan x)}{x}.$$

This is of indeterminate type $0/0$.

$$\begin{aligned}\lim_{x\to0+}\frac{d[\ln(e^x-2\arctan x)]/dx}{dx/dx} &= \lim_{x\to0+}\frac{1}{e^x-2\arctan x}\left(e^x-\frac{2}{1+x^2}\right)\\ &= \frac{1}{e^0-2(0)}\left(e^0-\frac{2}{1+0^2}\right)=-1.\end{aligned}$$

Hence, by l'Hospital's rule,

$$\lim_{x\to0+}\frac{\ln(e^x-2\arctan x)}{x}=-1.$$

Answer:

$$\lim_{x\to0+}(e^x-2\arctan x)^{1/x}=e^{-1}.$$

2.

$$\begin{aligned}\lim_{x\to\infty}(\sinh 2x-e^x\cosh x) &= \lim_{x\to\infty}\left(\frac{e^{2x}-e^{-2x}}{2}-e^x\frac{e^x+e^{-x}}{2}\right)\\ &= \lim_{x\to\infty}\frac{-e^{-2x}-1}{2}=\frac{-0-1}{2}=-\frac{1}{2}.\end{aligned}$$

3.

$$\begin{aligned}\frac{2x^3+8x-1}{x^3+x^2-x-1} &= 2-\frac{2x^2-10x-1}{x^3+x^2-x-1}=2-\frac{2x^2-10x-1}{(x-1)(x+1)^2}\\ &= 2+\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}\end{aligned}$$

for some numbers A, B and C.

4. (a) Integrate by parts with

$$u=\sec^{-1}\sqrt{x}\quad\text{and}\quad dv=x\,dx.$$

So

$$du = \frac{1}{2x\sqrt{x-1}}\,dx \quad \text{and} \quad v = \frac{x^2}{2}.$$

Then

$$\begin{aligned}
\int x \sec^{-1}\sqrt{x}\,dx &= \sec^{-1}\sqrt{x}\cdot\frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{1}{2x\sqrt{x-1}}\,dx \\
&= \frac{x^2}{2}\sec^{-1}\sqrt{x} - \frac{1}{4}\int \frac{x}{\sqrt{x-1}}\,dx \\
&= \frac{x^2}{2}\sec^{-1}\sqrt{x} - \frac{1}{4}\int\left(\sqrt{x-1} + \frac{1}{\sqrt{x-1}}\right)dx \\
&= \frac{x^2}{2}\sec^{-1}\sqrt{x} - \frac{1}{4}\left[\frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2}\right] + C \\
&= \frac{x^2}{2}\sec^{-1}\sqrt{x} - \frac{1}{6}(x-1)^{1/2}(x+2) + C.
\end{aligned}$$

(b) Split the integral into three parts.

$$\begin{aligned}
\int \tan^3 x \sec^5 x\,dx &= \int[(\sec^2 x - 1)\sec^4 x]\sec x \tan x\,dx = \int (u^2-1)u^4\,du \\
&= \int (u^6 - u^4)\,du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C \quad \text{where } u = \sec x.
\end{aligned}$$

$$\begin{aligned}
\int(-\tan^2 x \sin x)\,dx &= \int\left(-\frac{1-\cos^2 x}{\cos^2 x}\sin x\right)dx = \int \frac{1-u^2}{u^2}\,du \\
&= \int (u^{-2} - 1)\,du = -u^{-1} - u + C \quad \text{where } u = \cos x.
\end{aligned}$$

$$\int 4\sin^2 x\,dx = \int 2(1-\cos 2x)\,dx = 2x - \sin 2x + C.$$

Answer:

$$\begin{aligned}
&\int(\tan^3 x \sec^5 x - \tan^2 x \sin x + 4\sin^2 x)\,dx \\
&= \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x - \sec x - \cos x + 2x - \sin 2x + C.
\end{aligned}$$

(c) Completing the square, $x^2 - 6x + 5 = (x-3)^2 - 4$. Substitute $x - 3 = 2\sec\theta$. So $\sqrt{x^2-6x+5} = 2\tan\theta$, $x = 3 + 2\sec\theta$, and $dx = 2\sec\theta\tan\theta\,d\theta$. The integral becomes

$$\begin{aligned}
\int \frac{2x-1}{\sqrt{x^2-6x+5}}\,dx &= \int \frac{2(3+2\sec\theta)-1}{2\tan\theta}\cdot 2\sec\theta\tan\theta\,d\theta \\
&= \int (5\sec\theta + 4\sec^2\theta)\,d\theta \\
&= 5\ln|\sec\theta + \tan\theta| + 4\tan\theta + C.
\end{aligned}$$

Redefining the constant of integration gives

$$\int \frac{2x-1}{\sqrt{x^2-6x+5}}\,dx = 5\ln\left|x - 3 + \sqrt{x^2-6x+5}\right| + 2\sqrt{x^2-6x+5} + C.$$

(d) The partial fraction decomposition of the integrand is

$$\frac{A}{x-2} + \frac{Bx+C}{x^2+2x+10}$$

for some numbers A, B and C. Necessarily,

$$\begin{aligned} x^2+14 &= A(x^2+2x+10) + (Bx+C)(x-2) \\ &= (A+B)x^2 + (2A-2B+C)x + 10A - 2C. \end{aligned}$$

Equating coefficients gives

$$\begin{cases} A+B=1 \\ 2A-2B+C=0 \\ 10A-2C=14 \end{cases} \implies \begin{cases} A=1 \\ B=0 \\ C=-2. \end{cases}$$

So the integral becomes

$$\begin{aligned} \int \left(\frac{1}{x-2} - \frac{2}{x^2+2x+10} \right) dx &= \ln|x-2| - \int \frac{2}{(x+1)^2+9}\, dx \\ &= \ln|x-2| - \frac{2}{3} \arctan\left(\frac{x+1}{3} \right) + K. \end{aligned}$$

(e) Substitute $x = u^{-12}$. So $dx = -12u^{-13}\, du$. Then

$$\begin{aligned} \int \frac{1}{x\sqrt{\sqrt[6]{x}-1}}\, dx &= \int \frac{-12u^{-13}}{u^{-12}\sqrt{u^{-2}-1}}\, du = 12 \int \frac{-1}{\sqrt{1-u^2}}\, du \\ &= 12 \arccos u + C = 12 \arccos(x^{-1/12}) + C. \end{aligned}$$

An alternative answer is $-12 \arcsin u + C = -12 \arcsin(x^{-1/12}) + C$. Alternative substitutions are $x = u^{12}$ leading to $12 \sec^{-1} u + C$ or $-12 \csc^{-1} u + C$ with $u = \sqrt[12]{x}$, or, $x = (u^2+1)^6$ leading to $12 \arctan u + C$ or $-12 \cot^{-1} u + C$ with $u = \sqrt{\sqrt[6]{x}-1}$.

(f) Use the Weierstrass substitution $t = \tan(x/2)$. So $\sin x = 2t/(1+t^2)$ and $dx = [2/(1+t^2)]\, dt$. Then

$$\begin{aligned} \int \frac{1+2\tan(x/2)}{\sin x + 2}\, dx &= \int \frac{1+2t}{2t/(1+t^2)+2} \cdot \frac{2}{1+t^2}\, dt = \int \frac{2t+1}{t^2+t+1}\, dt \\ &= \ln(t^2+t+1) + C \\ &= \ln[\tan^2(x/2) + \tan(x/2) + 1] + C. \end{aligned}$$